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Non-commutative geometry and covariance: from the quantum plane to quantum tensors*

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Abstract

Reflection and braid equations for rank two q-tensors are derived from the covariance properties of quantum vectors by using the R-matrix formalism.

1 Introduction

Quantum groups may be looked at in various ways. From a mathematical point of view, they may be introduced by making emphasis on their q-deformed enveloping algebra aspects [1, 2] or by making emphasis in the R-matrix formalism that describe the deformed group algebra [3]. A point of view which is particularly useful in possible physical applications is to look at quantum groups as symmetries of *quantum spaces* [4, 5]. The simplest example of this approach is constituted by the well known quantum plane C_q^2, or associative *algebra* (a q-plane is not a manifold) generated by two elements $(x, y) = X$ (a q-two-vector) subjected to the commutation property [4]

$$xy = qyx \quad . \tag{1}$$

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The commutation relation (1) can also be expressed by using the q-symplectic metric ϵ^q

$$\epsilon^q = \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad , \qquad (\epsilon^q)^2 = -I \tag{2}$$

by the equation

$$X^t \epsilon^q X = 0 \quad , \qquad \epsilon^q_{ij} X_i X_j = 0 \tag{3}$$

which reflects that the q-symplectic norm of a q-two-vector vanishes.

It is also possible to introduce a pair of (odd) variables $(\xi, \eta) = \Omega$ (an odd q-two-vector) satisfying

$$\xi\eta = -\frac{1}{q}\eta\xi \quad , \qquad \xi^2 = 0 = \eta^2 \quad . \tag{4}$$

If it is required that, after the transformations $X' = TX$, $\Omega' = T\Omega$ the new entities (x', y'), (ξ', η') satisfy also (1), (4), it is found that the commutation properties of the elements of T

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{5}$$

are completely determined. These are the well known relations (the entries of T commute with those of X and Ω)

$$\begin{array}{llllll} ab = qba & , & ac = qca & , & bd = qdb & , \\ cd = qdc & , & [a,d] = \lambda cb & , & [b,c] = 0 & , \end{array} \tag{6}$$

($\lambda \equiv q - q^{-1}$) which constitute a presentation of the $GL_q(2, C)$ algebra generated by (a, b, c, d). For q=1, (x, y) commute and (ξ, η) anticommute. In a non-commutative differential calculus this second set of variables are identified with the differentials ([5]) of (x, y). Here we shall consider the quantum plane (1) only as the representation (co-module) space of the $GL_q(2, C)$ quantum group (6). In terms of the R-matrix formalism [3], eqs. (1) and (6) may be written as (see eq. (11) below)

$$R_{12} X_1 X_2 = q X_2 X_1 \quad \Longleftrightarrow \quad R_{21}^{-1} X_1 X_2 = q^{-1} X_2 X_1 \quad , \tag{7}$$

$$R_{12} T_1 T_2 = T_2 T_1 R_{12} \quad \Longleftrightarrow \quad R_{21}^{-1} T_1 T_2 = T_2 T_1 R_{21}^{-1} \quad , \tag{8}$$

where the standard notation $T_1 = T \otimes \mathbf{1}$, $T_2 = \mathbf{1} \otimes T$ $\;(T_{1ij,kl} = T_{ik}\delta_{jl}$, $T_{2ij,kl} = \delta_{ik}T_{jl}$, $i, j, k, l = 1, 2)$ has been used, and X_1X_2 and X_2X_1 are, respectively, the four-vectors (xx, xy, yx, yy) and (xx, yx, xy, yy). Both relations in (7) (and in (8)) are equivalent. This is easy to see by using the permutation operator $\mathcal{P}$ which gives $(\mathcal{P}R\mathcal{P})_{ij,kl} = R_{ji,lk}$ $(\mathcal{P}R_{12}\mathcal{P} = R_{21})$ and $(\mathcal{P}X_1X_2)_{ij} = (X_1X_2)_{ji}$ *i.e.*, $\mathcal{P}X_1X_2 = X_2X_1$. In this matrix notation it is obvious that (8) is consistent with the requirement of invariance of (7) under the transformation $X' = TX$,

$R_{12}X'_1X'_2 = qX'_2X'_1$. Since the elements of X commute with the entries of T, we obtain

$$\begin{aligned} R_{12}X'_1X'_2 &= R_{12}(T_1X_1)(T_2X_2) = R_{12}T_1T_2X_1X_2 \\ &= T_2T_1R_{12}X_1X_2 = qT_2T_1X_2X_1 = qX'_2X'_1 \end{aligned} \tag{9}$$

using (8), and the invariance of (7) follows: the preservation of (7) under the 'q-symmetry' transformation requires (8). In components, (7) reads

$$R_{ij,kl}X_kX_l = qX_jX_i \quad , \qquad \hat{R}_{ij,kl}X_kX_l = qX_iX_j \quad , \tag{10}$$

where R_{12} and $\mathcal{P}$ ($\hat{R} = \mathcal{P}R$, $\hat{R}_{ij,kl} = R_{ji,kl}$) are given by

$$R = \begin{bmatrix} q & & & \\ & 1 & 0 & \\ & \lambda & 1 & \\ & & & q \end{bmatrix} \quad , \quad \mathcal{P} = \begin{bmatrix} 1 & & & \\ & 0 & 1 & \\ & 1 & 0 & \\ & & & 1 \end{bmatrix} \quad . \tag{11}$$

Although the indices in all previous expressions take the values $1, 2$, the R-matrix form of the basic expressions (7) and (8) makes it clear how to generalize them to $GL_q(n, C)$; all that is needed is the appropriate $n^2 \times n^2$ R-matrix, which is given by [3]

$$R_{ij,kl} = \delta_{ik}\delta_{jl}(1 + \delta_{ij}(q-1)) + \lambda\delta_{il}\delta_{jk}\theta(i-j) \quad i, j... = 1...n \tag{12}$$

$$\theta(i-j) = \begin{cases} 0 & i \leq j \\ 1 & i > j \end{cases} \quad .$$

With it, the relations defining the 'quantum hyperplane'

$$X = (x_1, ..., x_n) \quad , \quad x_ix_j = qx_jx_i \quad (i < j)\; i, j = 1...n \tag{13}$$

are again expressed by (7) and preserved under $GL_q(n, C)$ because of (8).

All this is, of course, well known. In this report we exhibit how to extend these q-vector constructions to higher rank quantum tensors (see also [6, 7]). In particular, we shall consider the simplest example of q-twistors constructed from q-two-vectors (spinors) (1), (3), (7) and the application to q-Minkowski space algebras [8].

2 Other covariant objects. Quantum twistors

Consider two isomorphic objects X and Z, and their hermitian conjugates $X^\dagger$ and $Z^\dagger$, transforming under the coaction of two different quantum groups T and $T^\dagger$ by

$$\begin{array}{llll} X' = TX & , & X'^\dagger = X^\dagger T^\dagger & , \\ Z' = TZ & , & Z'^\dagger = Z^\dagger T^\dagger & , \end{array} \tag{14}$$

For instance, in the classical $SL(2, C)$ case there are two fundamental representations, $D^{\frac{1}{2},0}$ and $D^{0,\frac{1}{2}}$, realized by complex unimodular matrices A and

$(A^{-1})^\dagger$. In the quantum case this corresponds to taking two copies T and $\tilde{T}$ of $SL_q(2,C)$, with the obvious 'reality' condition added, $\tilde{T}^{-1}=T^\dagger$. In the q-case one has to add the commutation relations between elements of T and $T^\dagger$.

The commutation relations in the general situation involve four R-matrices $R^{(i)}$, $i=1,...,4$,

$$
\begin{aligned}
R^{(1)}T_1T_2 &= T_2T_1R^{(1)} \quad , \\
T_1^\dagger R^{(2)}T_2 &= T_2R^{(2)}T_1^\dagger \quad , \\
T_2^\dagger R^{(3)}T_1 &= T_1R^{(3)}T_2^\dagger \quad , \\
R^{(4)}T_1^\dagger T_2^\dagger &= T_2^\dagger T_1^\dagger R^{(4)} \quad .
\end{aligned}
\tag{15}
$$

The consistency of these equations requires $R^{(2)}=\mathcal{P}R^{(3)}\mathcal{P}=R^{(3)\,\dagger}$ and $R^{(4)}=R^{(1)\dagger}$ or $R^{(4)}=(\mathcal{P}R^{(1)\,-1}\mathcal{P})^\dagger$. Notice that $R^{(1)}$ and $R^{(4)}$ are the R-matrices of two quantum groups T and $T^\dagger$ related by a $*$-operation (and that $R^{(1)}$, *e.g.*, may be taken as R_{12} or R_{21}^{-1}). In contrast, $R^{(2)}$ (and hence $R^{(3)}$) is a matrix defining how the elements of both quantum groups commute and accordingly it is not a priori fixed. In general, one could introduce instead of $T^\dagger$ another matrix S; the q-matrices S and T need not even have the same dimension. If, say, T and S are $n\times n$ and $m\times m$ matrices, S_1 and T_2 in the second equation of (15) would be $S_1^\dagger=(S^\dagger\otimes\mathbf{1}_n)$ and $T_2=(\mathbf{1}_m\otimes T)$ and $R^{(2)}$ would be an $(m\times n)\times(m\times n)$ matrix. Similarly, in the third equation $S_2^\dagger=(\mathbf{1}_n\otimes S^\dagger)$, $T_1=(T\otimes\mathbf{1}_m)$ and $R^{(3)}$ would be an $(n\times m)\times(n\times m)$ matrix, while $R^{(4)}$ would be an $m^2\times m^2$ matrix.

The form of the eqs. (15) is the result of the equations which express the commutation relations among the components of the vectors X,Z. Since in principle T and $T^\dagger$ do not commute, we have to allow for possibly non-trivial commutation relations among the components of X and $Z^\dagger$. Thus, the set of commutation relations left invariant is given by

$$
\begin{aligned}
R^{(1)}X_1X_2 &= \kappa_1X_2X_1 \quad , \\
Z_1^\dagger R^{(2)}X_2 &= X_2Z_1^\dagger \quad , \\
Z_2^\dagger R^{(3)}X_1 &= X_1Z_2^\dagger \quad , \\
\kappa_2Z_1^\dagger Z_2^\dagger &= Z_2^\dagger Z_1^\dagger R^{(4)} \quad ,
\end{aligned}
\tag{16}
$$

where κ_1 and κ_2 are appropriate eigenvalues of the R-matrices. The invariance of the first and last equations is proven as in Sec.1 and the others similarly. For instance, for the second equation we check that

$$
Z_1'^\dagger R^{(2)}X_2' = (Z_1^\dagger T_1^\dagger)R^{(2)}(T_2X_2) = Z_1^\dagger T_2R^{(2)}T_1^\dagger X_2
$$

$$= T_2 Z_1^\dagger R^{(2)} X_2 T_1^\dagger = T_2 X_2 Z_1^\dagger T_1^\dagger = X_2' Z_1'^\dagger \tag{17}$$

using the second equations in (15) and (16), respectively, in the second and fourth equalities. In particular, if $R^{(2)} = I = R^{(3)}$, both quantum groups are independent (commuting), and this is reflected in the fact that the components of X and $Z^\dagger$ commute.

Let us use the above construction to introduce another covariant object which generalizes (with some restrictions) the concept of twistor to the q-deformed case. Let X and $Z^\dagger$ satisfy the previous set of commutation relations. In particular, X and $Z^\dagger$ may be, for instance, q-two-vectors (q-spinors), of $SL_q(2,C)$; this case will be analyzed in more detail below. Tensoring two q-vectors we introduce the object

$$K \equiv XZ^\dagger \qquad (K_{ij} = X_i Z_j^\dagger) \quad . \tag{18}$$

Then, the transformation of K induced by (14) is

$$\varphi : \; K \longmapsto K' = TKT^\dagger \qquad (K'_{ij} = T_{im} K_{mn} T^\dagger_{nj}) \quad . \tag{19}$$

The entries of K are, of course, non-commuting. We shall see that these commutation relations can be expressed in a closed, elegant and compact equation which permits to extract the algebra generated by the entries of K without considering its explicit realization in terms of the components of X and $Z^\dagger$. Using the above relations we may now derive the equation describing the commutation relations which define the algebra generated by the entries of K. With $K_1 = X_1 Z_1^\dagger$ ($K_{1\,ij,kl} = (K \otimes \mathbf{1})_{ij,kl} = X_i Z_k^\dagger \delta_{jl}$) and $K_2 = X_2 Z_2^\dagger$ ($K_{2\,ij,kl} = (\mathbf{1} \otimes K)_{ij,kl} = \delta_{ik} X_j Z_l^\dagger$), we find using (16) that

$$\begin{aligned} R^{(1)} K_1 R^{(2)} K_2 &= R^{(1)} X_1 Z_1^\dagger R^{(2)} X_2 Z_2^\dagger = R^{(1)} X_1 X_2 Z_1^\dagger Z_2^\dagger \\ &= (\kappa_1/\kappa_2) X_2 X_1 Z_2^\dagger Z_1^\dagger R^{(4)} = (\kappa_1/\kappa_2) X_2 Z_2^\dagger R^{(3)} X_1 Z_1^\dagger R^{(4)} \quad . \end{aligned} \tag{20}$$

Hence, the commuting properties of the quantum twistor are given by

$$R^{(1)} K_1 R^{(2)} K_2 = (\kappa_1/\kappa_2) K_2 R^{(3)} K_1 R^{(4)} \tag{21}$$

Eq. (21) is (with $\kappa_1/\kappa_2 = 1$) nothing else than the reflection equation with no spectral parameter dependence (see [6, 7] and references therein and [9] in the context of braided algebras) which follows by imposing the invariance of the commuting properties of the entries of K by the coaction (19). As shown here, eq. (21) also follows from interpreting K as an object made out of two q-'vectors', in general not necessarily of the same dimension so that in general K is not a squared matrix.

Let X, Z be two q-two-vectors (spinors). Then, $K = XX^\dagger$ is a (null) *quantum twistor*: as we shall see, its quantum determinant ($det_q K$) is necessarily zero (as it is as well for $XZ^\dagger$). In contrast, the q-twistor $K = XZ^\dagger + ZX^\dagger$ has $det_q K \neq 0$.

Notice that, in general, there are four possibilities to write (21) (obviously, related in between) since there are two possibilities for $R^{(1)}$ and for $R^{(4)}$ in (15) and in (16) (see (7) and (8)). However, this freedom is reduced when covariant objects K constructed out of four vectors are considered since covariance requires to introduce commutation relations between Z and X and between $Z^\dagger$ and $X^\dagger$ using $R^{(1)}$ and $R^{(4)}$. Let us consider the hermitian matrix

$$K = XZ^\dagger + ZX^\dagger \tag{22}$$

(Z and X have the same number of components). To compute the commutation properties of K, the complete set of relations among X, Z, $X^\dagger$ and $Z^\dagger$ are required. Thus, besides (16), we need to introduce the following set of covariant relations

$$\begin{aligned} &R^{(1)}X_1Z_2 = Z_2X_1 \quad , \\ &Z_1^\dagger R^{(2)}Z_2 = Z_2Z_1^\dagger \quad , \\ &X_2^\dagger R^{(3)}X_1 = X_1X_2^\dagger \quad , \\ &X_1^\dagger Z_2^\dagger = Z_2^\dagger X_1^\dagger R^{(4)} \quad , \end{aligned} \tag{23}$$

the structure of which is again dictated from (15) by covariance. From the first and the last eqs. in (23) we obtain (supposing $R^{(1)}$ real)

$$R^{(4)} = (R^{(1)})^t \tag{24}$$

which implies that the eigenvalues are equal, $\kappa_1 = \kappa_2$. Then, the commutation relation for the entries of K (in matrix form) are easily computed using (16) and (23)

$$\begin{aligned} R^{(1)}K_1R^{(2)}K_2 &= R^{(1)}(X_1Z_1^\dagger + Z_1X_1^\dagger)R^{(2)}(X_2Z_2^\dagger + Z_2X_2^\dagger) \\ &= (X_2Z_2^\dagger + Z_2X_2^\dagger)R^{(3)}(X_1Z_1^\dagger + Z_1X_1^\dagger)R^{(4)} \\ &= K_2R^{(3)}K_1R^{(4)} \end{aligned} \tag{25}$$

where $R^{(1)} = R^{(4)\,t} = \mathcal{P}R^{(4)}\mathcal{P}$ and $R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ and Hecke's condition for the R-matrix has been used. Now, we have only one reflection equation for K, since the two possibilities for $R^{(1)}$ produce two equations which are identical after a similarity transformation with $\mathcal{P}$.

Notice that K in (22) is constructed from two parts, each one of them satisfying the same algebra relations (25):

$$K = K^{(1)} + K^{(2)} \quad , \quad K^{(1)} = XZ^\dagger \quad , \quad K^{(2)} = ZX^\dagger \quad . \tag{26}$$

These two pieces have specific commutation properties among themselves. Indeed, the (mixed) commutation relations (23) lead to the following non-commuting property between the matrices $K^{(1)}$ and $K^{(2)}$ (non-symmetric under the interchange of $K^{(1)}$ and $K^{(2)}$)

$$R^{(1)}K_1^{(1)}R^{(2)}K_2^{(2)} = K_2^{(2)}R^{(3)}K_1^{(1)}(\mathcal{P}R^{(4)}\mathcal{P})^{-1} \quad . \tag{27}$$

Here $R^{(4)} = R^{(1)\,t}$ and the two possibilities for $R^{(1)}$ produce two different equations for $K^{(1)}$ and $K^{(2)}$ which transform into each other by exchanging (1)↔(2) in $K^{(i)}$. Both had to be possible since $K = K^{(1)} + K^{(2)}$ is symmetric under this exchange. Equation (27), here obtained from the commutation relations (23), is known as 'braiding equation' [9]. The commutation properties among the elements of $K^{(1)}$ and $K^{(2)}$ are such that the sum of two objects satisfying (25) verifies also the same relation. Within this terminology, the 'mixed' eqs. (23) are the braiding relations for q-vectors.

From now on, we shall restrict ourselves to the two-dimensional case which will be useful in the application to q-Minkowski space [8]. We shall start by discussing the

q-determinant of K:

Let the quantum group matrices T and $T^\dagger$ be 2×2 matrices. There exists an invariant quadratic element from K, the q-determinant of K. It is defined by [6]

$$det_q K \, P_- \equiv P_- K_1 \hat{R}^{(3)} K_1 P_- \tag{28}$$

where $\hat{R}^{(3)} = \mathcal{P} R^{(3)}$ and P_- is the q-antisymmetrizer of the R-matrix corresponding to the quantum groups T and $T^\dagger$. The q-determinant of T and $T^\dagger$ are given by [3]

$$det_q T \, P_- = P_- T_1 T_2 \qquad , \qquad det_q T^\dagger \, P_- = T_2^\dagger T_1^\dagger P_- \tag{29}$$

and the projector P_- can be expressed in terms of the q-epsilon tensor (2)

$$P_{-\;ij,kl} = [2]^{-1} \epsilon^q_{ij} \epsilon^q_{kl} \quad , \qquad P_- = \frac{1}{[2]} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & q^{-1} & -1 & 0 \\ 0 & -1 & q & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad . \tag{30}$$

When $(det_q T)(det_q T^\dagger) = 1$, $det_q K$ is invariant under the coaction (19). Using the third eq. in (15) and (29)

$$\begin{aligned} det_q(TKT^\dagger) &= P_-(T_1 K_1 T_1^\dagger) \hat{R}^{(3)} (T_1 K_1 T_1^\dagger) P_- \\ &= P_- T_1 T_2 K_1 \hat{R}^{(3)} K_1 T_2^\dagger T_1^\dagger P_- \\ &= (det_q T)\,(det_q K)\,(det_q T^\dagger) \; . \end{aligned} \tag{31}$$

Thus, if $(det_q T)(det_q T^\dagger) = 1$ we obtain that $det_q(TKT^\dagger) = det_q K$. The centrality of $det_q K$ requires some YBE-like conditions on the $R^{(i)}$ $(i = 1, 2, 3, 4)$ matrices in (21).

Using the definition (28) and the R-matrix property $\hat{R}^{(3)}_{ab,cd} = R^{(3)}_{ba,cd}$, we can compute explicitly the q-determinant of K in the following realizations

1. for the matrix $K = XZ^\dagger$ (and hence for the q-twistor $K = XX^\dagger$)

$$\begin{array}{rcl} (det_q K) P_{-\ ij,kl} & = & P_{-\ ij,ab} K_{ac} \hat{R}^{(3)}_{cb,mn} K_{mp} P_{-\ pn,kl} \\ & \propto & \epsilon^q_{ij} \epsilon^q_{ab} X_a Z^\dagger_c R^{(3)}_{bc,mn} X_m Z^\dagger_p \epsilon^q_{pn} \epsilon^q_{kl} \\ & = & \epsilon^q_{ij} \epsilon^q_{ab} X_a X_b Z^\dagger_n Z^\dagger_p \epsilon^q_{pn} \epsilon^q_{kl} \\ & = & \epsilon^q_{ij} (X^t \epsilon^q X)(Z^t \epsilon^q Z)^\dagger \epsilon^q_{kl} = 0 \end{array} \tag{32}$$

since $(X^t \epsilon^q X) = 0 = (Z^t \epsilon^q Z)$. This reflects the well-known fact in non deformed twistor theory that twistors constructed out of two spinors determine null length vectors;

2. for the q-twistor $K = XZ^\dagger + ZX^\dagger$, a similar calculus to the previous one gives

$$\begin{array}{rcl} (det_q K) P_{-\ ij,kl} & \propto & \epsilon^q_{ij} \epsilon^q_{ab} (X_a Z^\dagger_c + Z_a X^\dagger_c) R^{(3)}_{bc,mn} (X_m Z^\dagger_p + Z_m X^\dagger_p) \epsilon^q_{pn} \epsilon^q_{kl} \\ & = & \epsilon^q_{ij} [(X^t \epsilon^q Z)(X^t \epsilon^q Z)^\dagger + (Z^t \epsilon^q X)(Z^t \epsilon^q X)^\dagger] \epsilon^q_{kl} \ \neq 0 \end{array} \tag{33}$$

then, to get twistors with non-null q-determinant we need four spinors in the definition of K (notice that X, $X^\dagger$, Z and $Z^\dagger$ are algebraically independent objects). If the scalar products $(X^t \epsilon^q Z)$ and $(Z^t \epsilon^q X)$ are central elements in the algebra generated by X, Z, $X^\dagger$ and $Z^\dagger$ the q-determinant of K is also central.

3 An application: q-Minkowski space

The classical construction of a Minkowski vector uses two (dotted and undotted) spinors,

$$K_{\alpha\dot\beta} = \xi_\alpha \xi_{\dot\beta} = (\sigma_\mu x^\mu)_{\alpha\dot\beta} \quad \alpha, \dot\beta = 1,2 \quad , \tag{34}$$

and $K'_{\alpha\dot\beta} = A_\alpha{}^{.\gamma} K_{\gamma\dot\delta} (\tilde{A}^{-1})^{\dot\delta}{}_{.\dot\beta}$, where A and $\tilde{A} = (A^{-1})^\dagger$ are the two fundamental representations of $SL(2,C)$. A q-deformation of the Lorentz group may be obtained [10]-[12] by replacing A and $\tilde{A}$ by two copies T and $\tilde{T}$ of $SL_q(2,C)$. Applying the pattern described above we now have two pairs of q-spinors

$$X \to X' = TX \qquad Z \to Z' = TZ \quad , \tag{35}$$

$$X^\dagger \to X'^\dagger = X^\dagger \tilde{T}^{-1} \qquad Z^\dagger \to Z'^\dagger = Z^\dagger \tilde{T}^{-1} \quad , \tag{36}$$

obviously, the reality condition $T^\dagger = \tilde{T}^{-1}$ must be considered to have that $(X)^\dagger = X^\dagger$, from which we may construct the following hermitian objects (q-twistors)

$$K = XX^\dagger \quad \text{or} \quad K = XZ^\dagger + ZX^\dagger \quad , \tag{37}$$

and find their transformation properties. When the reality condition $T^\dagger = \tilde{T}^{-1}$ is imposed, the coaction

$$K' = TK\tilde{T}^{-1} = TKT^\dagger \tag{38}$$

preserves the hermiticity property of K. Since, by assumption, T and $\tilde{T}$ are $SL_q(2,C)$ matrices, *i.e.*,

$$R_{12}T_1T_2 = T_2T_1R_{12} \quad , \quad R_{12}\tilde{T}_1\tilde{T}_2 = \tilde{T}_2\tilde{T}_1R_{12} \quad , \tag{39}$$

the first and last equations of (15) are fulfilled if

$$\begin{array}{llll} R^{(1)} = R_{12} & \text{or} & R_{21}^{-1} & (\kappa_1 = q \quad \text{or} \quad q^{-1}) \quad , \\ R^{(4)} = R_{21} & \text{or} & R_{12}^{-1} & (\kappa_2 = q^{-1} \quad \text{or} \quad q) \quad . \end{array} \tag{40}$$

Then, the basic relations which define the non-commutative algebra generated by the entries of K are given by eq. (21), which gives the following possibilities

$$R_{12}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R_{21} \quad . \tag{41}$$

$$R_{12}K_1R^{(2)}K_2 = q^2K_2R^{(3)}K_1R_{12}^{-1} \quad . \tag{42}$$

As we have already discussed the second possibility (42) is not valid for the twistor with four spinors (second expression in (37)) since it does not correspond to $R^{(1)} = R^{(4)\,t}$. However, it is easy to check that the algebra generated by the entries of K satisfying eq. (42) coincides with the algebra determined by (41) with the additional condition $det_qK = 0$. To see it, the following consequences of the eigenvalue decomposition of R ($\hat{R} \equiv \mathcal{P}R = qP_+ - q^{-1}P_-$) are useful

$$P_-\hat{R} = \hat{R}P_- = -q^{-1}P_- \quad , \quad P_-\hat{R}^{-1} = \hat{R}^{-1}P_- = -qP_- \quad , \tag{43}$$

$$q^2\hat{R}^{-1} = \hat{R} - (q^3 - q^{-1})P_- \quad . \tag{44}$$

Multiplying now eq. (42) by $P_-\mathcal{P}$ from the left and by $\mathcal{P}P_-$ from the right and using (43) we get

$$-q^{-1}P_-K_1R^{(2)}K_2\mathcal{P}P_- = -q^3P_-\mathcal{P}K_2R^{(3)}K_1P_- \tag{45}$$

as $\hat{R}^{(i)} = \mathcal{P}R^{(i)}$, $R^{(2)} = \mathcal{P}R^{(3)}\mathcal{P}$ and $K_1 = \mathcal{P}K_2\mathcal{P}$

$$(q^3 - q^{-1})P_-K_1\hat{R}^{(3)}K_1P_- = 0 \quad . \tag{46}$$

Thus, (if $q^4 \neq 1$) we obtain that $det_qKP_- = P_-K_1\hat{R}^{(3)}K_1P_- = 0$.

Now , using (44) the RE (42) can be expressed in the following way

$$R_{12}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R_{21} - (q^3 - q^{-1})P_-K_2R^{(3)}K_1P_-\mathcal{P} \tag{47}$$

and using the definition of the q-determinant

$$R_{12}K_1R^{(2)}K_2 = K_2R^{(3)}K_1R_{21} - (q^3 - q^{-1})\mathcal{P}(det_qK)P_-\mathcal{P} \tag{48}$$

as $det_q K = 0$, we just obtained eq. (44); thus, as the algebra is the same eq. (42) may be discarded.

The matrices $R^{(2)}, R^{(3)}$ are not determined, and characterize the mixed commutation relations between quantum group elements and conjugated elements in (16), (23) and

$$\tilde{T}_1^{-1} R^{(2)} T_2 = T_2 R^{(2)} \tilde{T}_1^{-1} \quad , \quad T_1 R^{(3)} \tilde{T}_2^{-1} = \tilde{T}_2^{-1} R^{(3)} T_1 \ . \tag{49}$$

Two particularly special cases arise

a) Commuting case: if the two quantum group copies are independent, the quantum matrices commute

$$T_1 \tilde{T}_2 = \tilde{T}_2 T_1 \quad , \tag{50}$$

here, $R^{(2)} = R^{(3)} = I$, and then, eq. (41) gives the reflection equation, which is equivalent to the 'RTT' relation (18) (see below)

$$R_{12} K_1 K_2 = K_2 K_1 R_{21} \tag{51}$$

Eq. (42), in this particular case, produces the RE

$$R_{12} K_1 K_2 = q^2 K_2 K_1 R_{12}^{-1} \quad , \tag{52}$$

however, as we have just shown, this possibility leads to the same commutation relations (51) for the entries of K plus the additional condition $det_q K = 0$ [8]. The q-Minkowski algebra (51) is isomorphic to the quantum group algebra $GL_q(2)$, by [8]

$$T = K\sigma^1 \quad , \qquad R_{12} T_1 T_2 = T_2 T_1 R_{12} \quad , \tag{53}$$

where σ^1 is the usual Pauli matrix. Then, it is not possible to define a linear central element in the algebra generated by the entries $(\alpha, \beta, \gamma, \delta)$ of matrix K, and the quadratic one is given by the q-determinant (28) with $\hat{R}^{(3)} = \mathcal{P} R^{(3)} = \mathcal{P}$

$$det_q K P_- = P_- K_1 \mathcal{P} K_1 P_- = (-q^{-1})(\alpha\delta - q\gamma\beta) P_- \quad . \tag{54}$$

b) Non-commuting case: now, assuming the non-trivial commutation relations between the two copies of $SL_q(2, C)$

$$R_{12} T_1 \tilde{T}_2 = \tilde{T}_2 T_1 R_{12} \tag{55}$$

we see that (49) is fulfilled for $R^{(2)} = R_{21}$. Then, eq. (44) leads to the RE

$$R_{12} K_1 R_{21} K_2 = K_2 R_{12} K_1 R_{21} \tag{56}$$

Again, (42) produces an equation

$$R_{12} K_1 R_{21} K_2 = q^2 K_2 R_{12} K_1 R_{12}^{-1} \tag{57}$$

which leads to the same commutation relations as (56) with the restriction $det_q K = 0$.

These equations [8] define the quantum Minkowski algebra of [10]-[12], in which the linear central term is identified with the time coordinate and the q-determinant, defined by (28) where $\hat{R}^{(3)} = \hat{R}$

$$det_q K P_- = P_- K_1 \hat{R} K_1 P_- = (-q^{-1})(\alpha\delta - q^2 \beta\gamma) P_- \quad , \tag{58}$$

gives the quadratic central element which is identified with the invariant q-Minkowski length.

Having a q-vector $X \mapsto TX$ and a q-matrix $K \mapsto TKT^\dagger$, it is natural to construct higher rank tensors transforming as

$$\varphi : \ L \longmapsto T^{\otimes n} L (T^\dagger)^{\otimes n} \quad ; \tag{59}$$

they are invariant subspaces of the q-Minkowski algebra for the coaction φ. The generators of the q-tensors L may be extracted from matrices of higher dimensions, *e.g.*

$$\begin{gathered} L^2 \sim K^{\otimes_q 2} = K_1 R_{21} K_2 \quad \longmapsto T_1 T_2 K^{\otimes_q 2} T_1^\dagger T_2^\dagger \quad , \\ L^3 \sim K^{\otimes_q 3} = K_1 R_{21} K_2 R_{31} R_{32} K_3 \quad , \\ L^n \sim K^{\otimes_q n} = K_1 \prod_{j=2}^{n} (R_{j1} R_{j2} ... R_{j\, j-1} K_j) \quad . \end{gathered} \tag{60}$$

These subspaces (as in the non-deformed theory) are reducible (for instance, $K^{\otimes_q 2}$ has $det_q K$ as an invariant element). One can apply to $T^{\otimes 2} = T_1 T_2$ the appropriate projector $P^{(1)}$ to the spin 1 representation (and the same for $(T^\dagger)^{\otimes 2}$) to get a tensor of generators transforming according $D^{1,1}$ *irrep* of the q-Lorentz group. Quantum tensors transforming according $D^{j,s}$ *irreps* could be constructed in the same manner. We find additional R-matrix factors in the tensor products of K (60) (cf. (28)). This construction is useful for a description of higher spin q-wave equations (see also [13]).

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